THE FOKKER-PLANCK EQUATION FOR BOSONS IN 2D: WELL-POSEDNESS AND ASYMPTOTIC BEHAVIOR

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ABSTRACT. We show that solutions of the 2D Fokker-Planck equation for bosons are defined globally in time and converge to equilibrium, and this convergence is shown to be exponential for radially symmetric solutions. The main observation is that a variant of the Hopf-Cole transformation relates the 2D equation in radial coordinates to the usual linear Fokker-Planck equation. Hence, radially symmetric solutions can be computed analytically, and our results for general (non radially symmetric) solutions follow from comparison and entropy arguments. In order to show convergence to equilibrium we also prove a version of the Csiszár-Kullback inequality for the Bose-Einstein-Fokker-Planck entropy functional.

CONTENTS

1. INTRODUCTION

Introduced in the early 1900s [28, 16], Fokker-Planck equations have demonstrated to be a powerful tool to study macroscopic qualitative properties of systems of many interacting particles. They are particularly relevant in the field of statistical mechanics since the probability laws of Ornstein-Uhlenbeck or Langevin processes follow this type of partial differential equation [21, 29, 19, 17]. They are usually obtained as grazing collision limits of the Boltzmann equation or expansions from the master equation in classical [23, 12] and in quantum contexts [22, 24, 27, 25, 30, 2], and find applications in multiple areas [10, 6, 20]. In this way, nonlinear Fokker-Planck equations appear naturally as simplified kinetic equations with typically the same equilibrium states as the more physically grounded Boltzmann-type kinetic or master equations, but keeping some of the essential features and difficulties.

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In this paper, we work on the particular case of the model introduced in the nineties by Kaniadakis and Quarati [26] to describe the distribution of quantum particles in a spatially homogeneous setting leading to possible concentrations in finite time. From the physics point of view, it is quite relevant due to its relation to the Bose-Einstein condensate phenomena [14, 15, 31]. Different models have also been proposed to study this process, mostly based upon the Schrödinger equation and its Gross-Pitaevskii variants [1, 13, 32].

The basic idea is to introduce a nonlinear Fokker-Planck operator such that the Bose-Einstein distributions are their only possible steady states. These distributions are the family of functions of the form

$$f_\infty^\beta(v) := \frac{1}{\beta \exp(|v|^2/2) - 1}, \qquad v \in \mathbb{R}^d, \tag{1}$$

where $\beta \geq 1$ is a parameter. They obviously satisfy the equation

$$\nabla_v \log \left(\frac{f_\infty^\beta}{1 + f_\infty^\beta} \right)(v) + v = 0 \quad \Longleftrightarrow \quad \nabla_v f_\infty^\beta(v) + v f_\infty^\beta (1 + f_\infty^\beta)(v) = 0$$

for all $\beta > 1$. Clearly, for each $\beta > 1$ f_∞^β is a stationary solution to the following initial value problem for $f = f(t, v)$, with $t \geq 0$ and $v \in \mathbb{R}^d$:

$$\partial_t f = \Delta f + \text{div}\big(vf(1 + f)\big), \qquad t > 0, \ v \in \mathbb{R}^d, \tag{2a}$$

$$f(0, v) = f_0(v), \qquad v \in \mathbb{R}^d. \tag{2b}$$

The *mass* at time t of a solution f to eq. (2) is defined by its integral, and it is a conserved quantity of the time evolution, that is,

$$\int_{\mathbb{R}^d} f(t, v) \, dv = m := \int_{\mathbb{R}^d} f_0(v) \, dv , \qquad t > 0 .$$

Problem (2) is referred to as the Cauchy problem for the *Bose-Einstein-Fokker-Planck* (BEFP) equation or the *Fokker-Planck equation for bosons*. It is easy to check that the functions with $\beta > 1$ given by (1) are the only possible finite-mass smooth stationary solutions of (2).

In dimension $d \geq 3$ there are no stationary solutions with mass larger than that of f_∞^1, while in dimensions $d = 1, 2$ the mass of f_∞^1 is infinite and there is exactly one stationary solution for each fixed finite mass. We define the *critical mass* m_c by

$$m_c := \int_{\mathbb{R}^d} f_\infty^1(v) \, dv = \int_{\mathbb{R}^d} \frac{1}{\exp(|v|^2/2) - 1} \, dv, \tag{3}$$

which is finite in dimensions $d \geq 3$ and equal to ∞ if $d = 1, 2$. The mass of f_∞^β is actually explicitly computable in dimension $d = 2$:

$$\int_{\mathbb{R}^2} f_\infty^\beta(v) \, dv = 2\pi \log \left(\frac{\beta}{\beta - 1} \right) \quad \text{for } \beta > 1.$$

Since it is expected that solutions to (2) converge to an equilibrium with the same mass if available, one anticipates that solutions with mass $m \leq m_c$ are globally defined for all $t \in [0, \infty)$, while it is conjectured that solutions to (2) with mass $m > m_c$ will blow up at a certain finite time T^*.

Despite its simple formulation, very little can be said about the Cauchy problem for the BEFP equation (2) beyond the local existence of solutions [8, 7, 33], which can be proved by usual fixed-point arguments. It was proved by Toscani [33], via a contradiction argument involving the second moment, that in dimension $d \geq 3$ there are indeed certain solutions which blow up if the initial mass is sufficiently large or, for any given supercritical mass, if the initial second moment is small enough. The nature of the blow-up and its profile is an open question in $d \geq 3$. The numerical approach becomes then of even greater interest in clarifying the long time asymptotics of quantum models related to Bose-Einstein distributions, see for instance [34, 26]. We finally remark that in one dimension a direct minimization of the entropy functional for related equations can lead to singular parts in the equilibrium states [3].

We focus here on the 2-dimensional case, where the critical mass m_c is infinite and show that smooth solutions exist globally in time under certain mild conditions on the initial data. We always consider non-negative solutions to (2). One of the main difficulties that equation (2) presents when attempting to show the existence of global solutions is the lack of a priori estimates that would allow to control the L^p-norm of solutions for any $p > 1$.

Our analysis is based on the following change of variables, where, for the sake of clarity we shall omit, in general, the dependence on t and r or v of the different quantities that we will introduce. Consider a radially symmetric smooth solution $f(t, v) = |v|^{1-d} \varphi(t, |v|)$ to equation (2), with finite mass $m = \|f_0\|_{L^1} > 0$. Then $\varphi = \varphi(t, r)$ satisfies, for $t, r > 0$,

$$\partial_t \varphi = \partial_r^2 \varphi + \partial_r \left(\left(r - \frac{d-1}{r} \right) \varphi \right) + \partial_r \left(\frac{1}{r^{d-2}} \varphi^2 \right). \tag{4}$$

We notice that, since f is bounded at $v = 0$, φ satisfies that $\lim_{r \to 0} r^{-k} \varphi(t, r) = 0$ for all $k < d - 1$. Since f is radially symmetric and smooth, $\nabla f(t, 0) = 0$ and φ also satisfies the boundary condition

$$\partial_r \left(r^{1-d} \varphi \right) = 0 \quad \text{at } r = 0.$$

Expanding the derivative one sees that in particular $\lim_{r \to 0} \partial_r \varphi(t, r) = 0$. We now define $\Phi(t, r)$ as the indefinite integral of φ:

$$\Phi(t, r) := \int_0^r \varphi(t, s) \, ds. \tag{5}$$

That is, $d\omega_d\Phi$ is the mass of f in a ball of radius r, where ω_d is the volume of the d-dimensional unit ball. One sees that Φ satisfies

$$
\begin{cases}
\partial_t\Phi = \partial_r^2\Phi + \left(r - \dfrac{d-1}{r}\right)\partial_r\Phi + r^{2-d}\,|\partial_r\Phi|^2, \\[2mm]
\Phi(t,\infty) = \dfrac{m}{d\omega_d}.
\end{cases}
\tag{6}
$$

From now on, fix the dimension $d = 2$. Then the coefficient of the last term on the right hand side of equation (6) becomes independent of r and we can use the Hopf-Cole transformation. We introduce the new quantity

$$
\Psi(t,r) = e^{\Phi(t,r)} - 1.
\tag{7}
$$

Then, equation (6) becomes:

$$
\begin{cases}
\partial_t\Psi = \partial_r^2\Psi + \left(r - \dfrac{1}{r}\right)\partial_r\Psi, \\[2mm]
\Psi(t,\infty) = e^{m/d\omega_d} - 1,
\end{cases}
\tag{8}
$$

in which the quadratic term has been absorbed by the second derivative of Ψ with respect to r. We finally introduce the change

$$
\psi(t,r) := \partial_r\Psi(t,r),
\tag{9}
$$

which leads us to

$$
\partial_t\psi = \partial_r^2\psi + \partial_r\left(\left(r - \dfrac{1}{r}\right)\psi\right).
\tag{10}
$$

At this point we only need to notice that equation (10) coincides with the radial form of the 2D linear Fokker-Planck (FP) equation, namely

$$
\partial_t g = \Delta g + \operatorname{div}(vg), \qquad t > 0, \ v \in \mathbb{R}^2,
\tag{11a}
$$

$$
g(0,v) = g_0(v), \qquad v \in \mathbb{R}^2.
\tag{11b}
$$

More specifically, if g is radially symmetric and satisfies (11), then the function $\psi = \psi(t,r)$ defined by $\psi(t,|v|) = |v|g(t,v)$ solves (10). Hence, all the existing theory for equation (11) [29, 18] can be used and translated back to the BEFP equation (2) through the different changes of variable. This gives in particular an analytic expression for radially symmetric solutions of (2) and directly yields an existence and uniqueness theory for the (BEFP) equation (2) in dimension 2, as well as information about the asymptotic behavior of its solutions.

The above changes of variable are summarized in the following diagram:

$$
f \xrightarrow{\ rf=\varphi(r)\ } \varphi \xrightarrow{\ \int_0^r \varphi\ } \Phi \xrightarrow{\ e^\Phi-1\ } \Psi \xrightarrow{\ \partial_r\Psi\ } \psi \xrightarrow{\ rg=\psi(r)\ } g
$$
$$
f = \Lambda(g)
$$

The rest of the paper is organized as follows. In Section 2 we make precise the translation between solutions to the Bose-Einstein-Fokker-Planck equation (2) and solutions to the linear Fokker-Planck equation (11). Next, in Section 3 we present our main results on radially symmetric solutions, Theorems 3.3 and 3.6. Then, in Section 4 we extend these results to more general solutions by comparison with radial solutions, and state our main results for non-radially symmetric solutions in Theorems 4.2 and 4.3. We state the Csiszár-Kullback inequality for the Bose-Einstein-Fokker-Planck entropy in Lemma 4.5, which is needed for the proof of Theorem 4.3.

2. The change of variables

We will now make precise the equivalence between radially symmetric solutions of the FP equation (11) and those of the BEFP equation (2). This will hold for classical solutions of BEFP, as given in the next definition. We denote by $\mathcal{C}_t^1 \mathcal{C}_v^2$ the space of functions $f = f(t, v)$ on $(0, \infty) \times \mathbb{R}^2$ for which f, $\partial_t f$, and all partial derivatives of order up to 2 with respect to v exist and are continuous on $(0, \infty) \times \mathbb{R}^2$. A *measure* is always understood to be a non-negative Borel measure with finite total mass and the space of such measures is denoted by $\mathcal{M}_+(\mathbb{R}^2)$.

Definition 2.1 (Classical solution to BEFP). Let f_0 be a nonnegative Borel measure in $\mathbb{R}^2$ (not necessarily finite). We say that a function f defined on $(0, \infty) \times \mathbb{R}^2$ is a *classical solution* to the Cauchy problem for the BEFD equation (2) with initial condition f_0 when $f = f(t, v)$ belongs to $\mathcal{C}_t^1 \mathcal{C}_v^2$, equation (2a) holds in $(0, \infty) \times \mathbb{R}^2$, and $\lim_{t \to 0} f(t) = f_0$ in the weak-$*$ sense of measures on $\mathbb{R}^2$.

We define a classical solution to the Cauchy problem for the FP equation (11) analogously. Note that classical solutions of the FP equation are known to be unique for measures with finite total mass as initial data via the change of variables to the heat equation, see next section and [9]. However, since our change of variables works for locally finite measures, we prefer to state the concept in this more general setting.

The change of variables (5), (7), and (9) can be summarized as follows: the non-negative, radially symmetric functions f and g on $\mathbb{R}^2$ are related by

$$|v| f(t, v) = \varphi(t, |v|), \quad |v| g(t, v) = \psi(t, |v|), \qquad t > 0, \ v \in \mathbb{R}^2, \qquad (12)$$

with

$$\varphi(t, r) := \partial_r \log \left(1 + \int_0^r \psi(t, s) \, \mathrm{d}s \right) \quad \text{for } t > 0, \ r \geq 0, \qquad (13)$$

or equivalently

$$\psi(t, r) := \partial_r \exp \left(\int_0^r \varphi(t, s) \, \mathrm{d}s \right) \quad \text{for } t > 0, \ r \geq 0. \qquad (14)$$

This is also equivalent to

$$\varphi(t,r) = \frac{\psi(t,r)}{1 + \int_0^r \psi(t,s)\, ds}, \qquad \psi(t,r) = \varphi(t,r) \exp\left(\int_0^r \varphi(t,s)\, ds\right), \qquad (15)$$

which may be a more convenient form for some calculations.

We may also adapt the change of variables (12)–(14) to work on locally finite non-negative measures (i.e., measures which are finite on any compact set), so that we can consider a measure as initial condition. If f and g are non-negative, locally finite, radially symmetric measures on $\mathbb{R}^2$ then one may define measures φ and ψ on $[0, \infty)$ by

$$\varphi([0,r]) := \frac{1}{2\pi} f\left(\overline{B(0,r)}\right), \quad \psi([0,r]) := \frac{1}{2\pi} g\left(\overline{B(0,r)}\right), \qquad r \geq 0. \qquad (16)$$

Conversely, if φ and ψ are radially symmetric measures on $[0, \infty)$, then there are unique radially symmetric measures f and g on $\mathbb{R}^2$ which satisfy (16). When f and g are functions, φ and ψ are just defined by (12).

The change (13)–(14) may be extended to non-negative, locally finite measures by setting

$$\varphi([0,r]) = \log\left(1 + \psi([0,r])\right) \qquad \text{for } r \geq 0, \qquad (17)$$

$$\psi([0,r]) = \exp\left(\varphi([0,r])\right) - 1 \qquad \text{for } r \geq 0. \qquad (18)$$

One can check that the change (16)–(18) coincides with (12)–(14) when f and g (and hence also φ and ψ) are locally integrable functions.

Hence, by setting $\Lambda(g) = f$ where f is related to g through φ and ψ, see equations (16)–(18), we define a transformation Λ on the set of non-negative, radially symmetric, locally finite measures on $\mathbb{R}^2$. We observe that from (17)–(18) one easily deduces that Λ and Λ^{-1} are continuous in the weak-$*$ topology of measures. We will often consider Λ restricted to the set $\mathcal{M}_{+,\mathrm{rad}}(\mathbb{R}^2)$ of non-negative, radially symmetric, finite measures on $\mathbb{R}^2$. In that set, the masses (total variation) of f (denoted by m) and g (denoted by M) are related by

$$m = 2\pi \log\left(1 + \frac{M}{2\pi}\right), \qquad M = 2\pi\left(e^{m/2\pi} - 1\right). \qquad (19)$$

The transformation Λ is also continuous in $L^1_{\mathrm{rad}}(\mathbb{R}^2)$, the set of integrable and radially symmetric functions in $\mathbb{R}^2$:

Lemma 2.2. *The change Λ is a bijection in $L^1_{rad}(\mathbb{R}^2)$, and both Λ and Λ^{-1} are locally Lipschitz continuous in the L^1-norm.*

Proof. That Λ is a bijection in $L^1_{\mathrm{rad}}(\mathbb{R}^2)$ is clearly seen from (15), for example. Take g_1 and g_2 in $L^1_{\mathrm{rad}}(\mathbb{R}^2)$ and set $f_i = \Lambda(g_i)$, $|v|\varphi_i(|v|) = f_i(v)$, and $|v|\psi_i(|v|) = g_i(v)$ for $i = 1, 2$ and $v \in \mathbb{R}^2$. Set $F_i(r) = (1 + \int_0^r \psi_i(s)\, ds)^{-1}$ for $i = 1, 2$ and notice that

$$0 \leq F_i(r) \leq 1 \quad \text{and} \quad |F_1(r) - F_2(r)| \leq \int_0^r |\psi_1(s) - \psi_2(s)|\, ds, \quad r \geq 0.$$

Combining these inequalities with (15) gives

$$
\begin{aligned}
\frac{1}{2\pi}\|f_1 - f_2\|_1 &= \int_0^\infty |\varphi_1(r) - \varphi_2(r)| \, \mathrm{d}r = \int_0^\infty |F_1(r)\psi_1(r) - F_2(r)\psi_2(r)| \, \mathrm{d}r \\
&\leq \int_0^\infty F_1(r)\,|\psi_1(r) - \psi_2(r)| \, \mathrm{d}r + \int_0^\infty |F_1(r) - F_2(r)|\,\psi_2(r) \, \mathrm{d}r \\
&\leq \int_0^\infty |\psi_1(r) - \psi_2(r)| \, \mathrm{d}r + \int_0^\infty \int_0^r |\psi_1(s) - \psi_2(s)|\,\psi_2(r) \, \mathrm{d}s \, \mathrm{d}r \\
&\leq \frac{1}{2\pi}\|g_1 - g_2\|_1 + \frac{1}{4\pi^2}\|g_1 - g_2\|_1\|g_2\|_1,
\end{aligned}
$$

which shows that Λ is locally Lipschitz continuous in the L^1-norm. A very similar argument shows that Λ^{-1} is also locally Lipschitz continuous. $\qquad\square$

This change of variables is also very regular, in the sense that it preserves the smoothness of the function to which it is applied. This will enable us later to translate properties of the FP equation (11) to the BEFP equation (2) in dimension 2:

Lemma 2.3. *The change Λ is a bijection from the set of radially symmetric functions in $\mathcal{C}^2(\mathbb{R}^2)$ to itself (in the sense that $\Lambda(g)$ coincides a.e. with a function in $\mathcal{C}^2(\mathbb{R}^2)$). In a similar way, if $g \in \mathcal{C}_t^1\mathcal{C}_v^2$ with $g(t)$ radially symmetric for each $t > 0$ and we define $f(t,v) = \Lambda(g(t))(v)$ then $f \in \mathcal{C}_t^1\mathcal{C}_v^2$.*

Before proving this we mention an elementary lemma that we wish to make explicit. We denote by $\mathcal{C}^2([0,\infty))$ the space of twice differentiable real functions on $[0,\infty)$ such that all derivatives of order up to 2 have limits at 0.

Lemma 2.4 (Regularity of a radial function). *Take $\phi \in \mathcal{C}^2([0,\infty))$ such that*

$$
\lim_{r \to 0} \phi'(r) = 0. \tag{20}
$$

If we define a radial function $f \colon \mathbb{R}^d \to \mathbb{R}$ by $f(v) = \phi(|v|)$, then $f \in \mathcal{C}^2(\mathbb{R}^d)$. Conversely, if $f \in \mathcal{C}^2(\mathbb{R}^d)$ is a radially symmetric function, then the function $\phi \colon [0,\infty) \to \mathbb{R}$ defined by $\phi(r) := f(r,0,\ldots,0)$ belongs to $\mathcal{C}^2([0,\infty))$ and satisfies (20).

Remark 2.5. It is easy to extend this lemma also to time-dependent functions: for example, if a function $\phi = \phi(t,r)$ is in $\mathcal{C}_t^1\mathcal{C}_r^2((0,\infty) \times [0,\infty))$ and satisfies

$$
\lim_{(s,r) \to (t,0)} \partial_r \phi(s,r) = 0
$$

for all $t > 0$, then $f(t,v) := \phi(t,|v|)$ is in $\mathcal{C}_t^1\mathcal{C}_v^2$.

Proof of Lemma 2.4. Clearly $f \in \mathcal{C}^2(\mathbb{R}^d \setminus \{0\})$ and we only need to check that the first and second-order derivatives of f can be extended continuously at $v = 0$. For

$1 \leq i, j \leq d$ and $v \in \mathbb{R}^d$, $v \neq 0$,

$$\partial_i f(v) = \phi'(|v|)\frac{v_i}{|v|} , \quad \partial_i \partial_j f(v) = \delta_{ij}\phi''(|v|) + \left(\frac{v_i v_j}{|v|^2} - \delta_{ij}\right)\left(\phi''(|v|) - \frac{\phi'(|v|)}{|v|}\right) .$$

Since $v \mapsto v_i/|v|$ and $v \mapsto v_i v_j/|v|^2$ are bounded and

$$\lim_{r \to 0}\left(\phi''(r) - \frac{\phi'(r)}{r}\right) = \phi''(0) - \phi''(0) = 0$$

by (20) we conclude that

$$\lim_{v \to 0}\partial_i f(v) = 0 \quad \text{and} \quad \lim_{v \to 0}\partial_i \partial_j f(v) = \phi''(0)\delta_{ij} .$$

Consequently, f can be extended to a $\mathcal{C}^2$-smooth function in $\mathbb{R}^d$ by setting $\nabla f(0) = 0$ and $D^2 f(0) = \phi''(0)\mathrm{id}$.

The converse is obvious as the $\mathcal{C}^2$-regularity of ϕ readily follows from that of f while the radial symmetry of f guarantees that $\nabla f(0) = 0$ and thus (20). □

Proof of Lemma 2.3. Take a radially symmetric function $g \in \mathcal{C}^2(\mathbb{R}^2)$ and let us show that $f = \Lambda(g) \in \mathcal{C}^2(\mathbb{R}^2)$ (the proof for Λ^{-1} is similar). Clearly f is $\mathcal{C}^2$-smooth in $\mathbb{R}^2 \setminus \{0\}$, since the change of variables has a smooth expression away from $v = 0$. In order to see that f is twice differentiable at $v = 0$, we use (12) and (15) to write f as

$$f(v) = \frac{\varphi(|v|)}{|v|} = \frac{\psi(|v|)}{|v|}\left(1 + \int_0^{|v|}\psi(s)ds\right)^{-1} = g(v)F(|v|),$$

where we have set

$$F(r) := \left(1 + \int_0^r \psi(s)ds\right)^{-1} , \quad r \geq 0.$$

Observe that $\psi(r) = rg(r, 0, \ldots, 0)$ for $r \geq 0$ and thus belongs to $\mathcal{C}^2([0, \infty))$ with $\psi(0) = 0$. then $F'(0) = 0$ and we deduce from Lemma 2.4 that f extends to a $\mathcal{C}^2$-smooth function in $\mathbb{R}^2$. The time-dependent results can be proved with very little modifications. □

We have the following result:

Theorem 2.6 (Equivalence between BEFP and FP). *Let f_0 and g_0 be non-negative radially symmetric measures on $\mathbb{R}^2$ related by (16)–(18), and let f, g be non-negative radially symmetric functions on $(0, \infty) \times \mathbb{R}^2$ related by (12)–(14). The following are equivalent:*

(1) *f is a classical solution to the Bose-Einstein-Fokker-Planck equation (2) with initial condition f_0.*

(2) *g is a classical solution to the linear Fokker-Planck equation (11) with initial condition g_0.*

Proof. We just show one of the implications, since the reverse one is analogous. Assume point (2), and let us show point (1). Since g is a classical solution to (11), it lies in $\mathcal{C}_t^1 \mathcal{C}_v^2$. Hence the calculations (4)–(10) are rigorous and we see that f satisfies (2) in $(0, \infty) \times (\mathbb{R}^2 \setminus \{0\})$ (the point $v = 0$ being a possible problem due to the radial change of variables.) But due to Lemma 2.3, f belongs to $\mathcal{C}_t^1 \mathcal{C}_v^2$ and a continuity argument entails that f satisfies (2a) in $(0, \infty) \times \mathbb{R}^2$. Finally, f satisfies the initial condition in the weak-$*$ sense since the change Λ is continuous for the weak-$*$ convergence of measures. $\qquad\square$

Notice that some of the classical solutions given by Theorem 2.6 are not covered by any of the existence results in the literature. The change of variables gives us classical solutions to a nonlinear equation with a measure as initial condition, which is quite remarkable. Finally, let us point out that radially symmetric classical solutions to the FP equation (11) are unique as soon as some growth condition at infinity is satisfied. Since the FP equation (11) is equivalent to the heat equation as we will see next, then uniqueness is satisfied for initial data in tempered distributions (classical solutions are distributional solutions), in particular for finite mass non-negative measures (which are our main interest).

3. Global radially symmetric solutions

The Fokker-Planck equation (11) is equivalent to the heat equation via a self-similar change of variables. Moreover, it can be explicitly solved through the Fourier transform and its fundamental solution can be explicitly found leading to

$$\mathcal{F}(t, v, w) := a(t)^{-d/2} M_{\nu(t)}(a(t)^{-1/2}v - w)$$

with

$$a(t) := \mathrm{e}^{-2t} \quad , \quad \nu(t) := \mathrm{e}^{2t} - 1 \quad \text{and} \quad M_\lambda(\xi) := (2\pi\lambda)^{-d/2}\mathrm{e}^{-|\xi|^2/2\lambda}$$

for any $\lambda > 0$. Let us focus again on the two dimensional case and define the operator $\mathcal{F}[\xi]$ acting on functions ξ by:

$$\mathcal{F}[\xi(w)](t, v) := \int_{\mathbb{R}^2} \mathcal{F}(t, v, w)\xi(w)\,\mathrm{d}w , \quad (t, v) \in (0, \infty) \times \mathbb{R}^2 . \qquad (21)$$

The following result is well-known, see for example [9, Lemma 3.1]:

Theorem 3.1 (Solutions to FP). *Let $S'(\mathbb{R}^2)$ denote the space of tempered distributions. Then, given $g_0 \in S'(\mathbb{R}^2)$ there exists a unique weak solution $g \in \mathcal{C}([0, \infty), S'(\mathbb{R}^2))$ to the linear Fokker-Planck equation (11) with initial condition g_0 given by $g = \mathcal{F}[g_0]$. In addition, g is smooth on $(0, \infty) \times \mathbb{R}^2$, and positivity and radial symmetry are conserved; more precisely:*

(1) *If g_0 is not identically equal to zero then $g(t, v) > 0$ for all $t > 0$ and $v \in \mathbb{R}^2$.*

(2) *If g_0 is radially symmetric then $g(t)$ is radially symmetric for all $t > 0$.*

Remark 3.2. Since $\mathcal{M}_+(\mathbb{R}^2)$ endowed with the weak-$*$ topology is continuously embedded in $S'(\mathbb{R}^2)$, in our context this result tells us that, indeed, for any $g_0 \in \mathcal{M}_+(\mathbb{R}^2)$ there exists a unique classical solution $g \in \mathcal{C}([0,\infty), \mathcal{M}_+(\mathbb{R}^2))$ of the FP equation (11) (where we always consider the weak-$*$ topology on $\mathcal{M}_+(\mathbb{R}^2)$) and this solution satisfies the above properties.

Finally, let us recall some moment and smoothing properties of the solution of the FP equation (11). With this aim, let us introduce the spaces $L^p_\ell(\mathbb{R}^2)$ with norms

$$\|f\|_{L^p_\ell} := \|(1 + |v|^\ell)f\|_p \quad \text{and} \quad \|f\|_p := \left(\int_{\mathbb{R}^d} |f|^p \mathrm{d}v \right)^{\frac{1}{p}}.$$

for $1 \le p \le \infty$ and $\ell \ge 0$, with the usual essential supremum norm definition for the case $p = \infty$. Solutions g to equation (11) satisfy

$$\|\partial_\alpha g(t)\|_{L^p_\ell} \le \frac{Ce^{\left(\frac{2(p-1)}{p} + |\alpha|\right)t}}{\nu(t)^{\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{|\alpha|}{2}}} \|g_0\|_{L^q_\ell},$$

for all $1 \le q \le p \le \infty$, $\ell \ge 0$, $\alpha \in \mathbb{N}^d$ a multiindex, and $t > 0$. Here, $C > 0$ is a number depending on p, q, ℓ and α. For a proof of this result see [7, Proposition A.1]. In particular, the case $\alpha = 0$ and $p = q$ gives

$$\|g(t)\|_{L^p_\ell} \le Ce^{\frac{2(p-1)t}{p}} \|g_0\|_{L^p_\ell} \tag{22}$$

for all $1 \le p \le \infty$, $\ell \ge 0$, and $t > 0$ with C depending only on p and ℓ. The case $\alpha = 0$, $q = 1$ gives

$$\|g(t)\|_{L^p_\ell} \le C \left(\frac{e^{2t}}{e^{2t} - 1} \right)^{(p-1)/p} \|g_0\|_{L^1_\ell}, \quad t > 0. \tag{23}$$

Theorem 2.6 immediately implies an analogous result for *radially symmetric* solutions to the BEFP equation (2):

Theorem 3.3 (Radial solutions to BEFP). *For any radially symmetric $f_0 \in \mathcal{M}_+(\mathbb{R}^2)$ there is a unique classical solution $f \in \mathcal{C}([0,\infty), \mathcal{M}_+(\mathbb{R}^2))$ to the BEFP equation (2) with initial condition f_0 and it is given by $f(t) = \Lambda(g(t))$ for all $t \ge 0$ where g is the solution to the FP equation (11) with initial condition $g_0 := \Lambda^{-1}(f_0)$. In addition, f is bounded in $L^\infty((\delta, \infty) \times \mathbb{R}^2)$ for each $\delta > 0$ and $f(t)$ is radially symmetric for all $t \ge 0$. Furthermore, if f_0 is not identically equal to zero then $f(t,v) > 0$ for all $t > 0$ and $v \in \mathbb{R}^2$.*

Remark 3.4. This solution is also the unique mild solution in the sense of [7, Eq. (2.2)] for initial data in the class considered therein, since the authors prove that mild solutions are unique up to the maximal existence time. Of course, the above theorem also shows that the unique mild solution does not blow up and is defined for all positive times.

Proof. Consider the solution g to the FP equation (11) with initial condition $\Lambda^{-1}(f_0)$ given by Theorem 3.1. This solution is also in $\mathcal{C}([0,\infty), \mathcal{M}_+(\mathbb{R}^2))$ (see Remark 3.2). Since g is a classical solution to the FP equation (11), the function $f := \Lambda(g)$ is a classical solution to the BEFP equation (2) by Theorem 2.6. It also lies in $\mathcal{C}([0,\infty), \mathcal{M}_+(\mathbb{R}^2))$, since Λ preserves this space and is continuous with respect to the weak-$*$ topology of measures. Finally, the claimed boundedness in $(\delta,\infty) \times \mathbb{R}^2$ readily follows from (23). $\qquad\square$

We next give another example of the results that one can get by translating known results for the FP equation (11), which deals with the large time behavior. The following result is well known and can be found for example in [9, Theorem 4.1].

Theorem 3.5 (Asymptotic behavior for FP). *Take $g_0 \in L^1_+(\mathbb{R}^2)$ and denote the total mass of g_0 by M. The solution g of the FP equation (11) with initial condition g_0 satisfies*

$$\|g(t) - Mg_\infty\|_{L^1} \le e^{-t}\|g_0 - Mg_\infty\|_{L^1} \quad \text{for all } t \ge 0,$$

where the equilibrium g_∞ is given by

$$g_\infty(v) = \frac{1}{2\pi}e^{-|v|^2/2}, \qquad v \in \mathbb{R}^2.$$

Notice that if we apply the change (12)–(13) to Mg_∞ we obtain precisely f^β_∞, with

$$\beta = \frac{2\pi}{M} + 1. \tag{24}$$

Thanks to Theorem 2.6 and Lemma 2.2, this translates into the following result for the BEFP equation (2)

Theorem 3.6 (Asymptotic behavior for radial BEFP). *Take a radially symmetric $f_0 \in L^1_+(\mathbb{R}^2)$ and denote its mass by m. There is a constant $K = K(m) > 0$, depending only on m, such that the solution f of the BEFP equation (2) with initial condition f_0 satisfies*

$$\|f(t) - f^\beta_\infty\|_{L^1} \le Ke^{-t}\|f_0 - f^\beta_\infty\|_{L^1} \quad \text{for all } t \ge 0,$$

choosing β to be such that f^β_∞ has mass m:

$$\beta = \left(1 - e^{-m/(2\pi)}\right)^{-1}. \tag{25}$$

Remark 3.7. The constant $K(m)$ above can be taken to be

$$K(m) = \left(\mathrm{Lip}_M(\Lambda)\right)\left(\mathrm{Lip}_m(\Lambda^{-1})\right),$$

where Lip_R denotes the L^1-Lipschitz constant in the L^1-ball of radius R (see Lemma 2.2.)

Another interesting consequence of Theorem 2.6 is that radially symmetric solutions to the BEFP equation (2) have an explicit expression in terms of the initial condition f_0, obtained by transforming the explicitly computable solution g of the FP equation (11) through the change of variables (12)–(15):

$$f(t,v) = \frac{g(t,v)}{1 + \int_0^{|v|} s\psi(t,s)\,\mathrm{d}s}, \quad (t,v) \in (0,\infty) \times \mathbb{R}^2 ,$$

with ψ related to g by (12). This relation obviously implies that, for $(t,v) \in (0,\infty) \times \mathbb{R}^2$,

$$\frac{2\pi}{2\pi + \|g_0\|_{L^1}} g(t,v) \le f(t,v) \le g(t,v) \le f(t,v) e^{\|f_0\|_{L^1}/2\pi}$$

by the conservation of the L^1-norm. This readily translates bounds for g into bounds for f; indeed, on the one hand, equation (22) gives

$$\|f(t)\|_{L_\ell^p} \le C e^{\frac{2(p-1)t}{p}} \|g_0\|_{L_\ell^p} \le C e^{\frac{2(p-1)t}{p}} e^{\|f_0\|_{L^1}/2\pi} \|f_0\|_{L_\ell^p} \tag{26}$$

for all $1 \le p \le \infty$, $\ell \ge 0$, and $t > 0$ with C depending only on p and ℓ. On the other hand, equation (23) gives

$$\begin{aligned}
\|f(t)\|_{L_\ell^p} &\le C\left(\frac{e^{2t}}{e^{2t}-1}\right)^{(p-1)/p} \|g_0\|_{L_\ell^1} \\
&\le C\left(\frac{e^{2t}}{e^{2t}-1}\right)^{(p-1)/p} e^{\|f_0\|_{L^1}/2\pi} \|f_0\|_{L_\ell^1}.
\end{aligned} \tag{27}$$

These bounds on f will be used in the next section in order to show that solutions to (2) starting from general inital data are globally defined in time.

An interesting particular solution is the transform of the solution to the FP equation (11) with initial condition the Dirac mass δ_0 at $v = 0$ which may be referred to as a *fundamental solution* for the (nonlinear) BEFP equation (2):

$$f(t,v) = \frac{1}{\vartheta(t)} \left[(2\pi+1) e^{|v|^2/2\vartheta(t)} - 1 \right]^{-1}, \quad \text{for } v \in \mathbb{R}^2, t \ge 0,$$

with

$$\vartheta(t) := (1 - e^{-2t}) \quad \text{for } t \ge 0.$$

There is also an explicit infinite-mass solution to the BEFP equation (2) obtained by transforming the solution $g(t,v) = Ae^{2t}$, $A > 0$, to the FP equation (11):

$$f(t,v) = 2\left(2A^{-1}e^{-2t} + |v|^2\right)^{-1}, \quad \text{for } v \in \mathbb{R}^2, t \ge 0.$$

Strictly speaking, this solution does not satisfy Definition 2.1 because it is not integrable; however, it satisfies (2a) for all $(t,v) \in (0,\infty) \times \mathbb{R}^2$. This solution converges exponentially fast to an infinite-mass stationary solution $f_\infty^*(v) := |v|^{-2}$.

4. Global (non-radially symmetric) solutions

In the case of Fermi-Dirac interactions, where the nonlinear drift in (2) is negative, a local well-posedness theory and L^1-contraction and maximum principles were proved in [7]. However, the proofs are totally independent of the sign of the drift and can be directly translated to our case:

Proposition 4.1 (Summary of results from [7])**.** *Let $f_0 \in (L^1 \cap L^p_\ell)(\mathbb{R}^2)$ be non-negative, with $p > 2$ and $\ell \geq 1$. Then:*

> **Local Existence:** *There exists a maximal time $T \in (0, \infty]$ depending only on $\|f_0\|_{L^p_\ell}$ and $\|f_0\|_{L^1}$ such that there is a unique solution f in $\mathcal{C}([0,T), (L^p_\ell \cap L^1)(\mathbb{R}^2))$ of the BEFP equation (2) which is non-negative and satisfies*
> $$\|f(t)\|_{L^1} = \|f_0\|_{L^1} , \qquad t \in [0,T) .$$
> *If $T < \infty$ then $\|f(t)\|_{L^p_\ell} \longrightarrow \infty$ as $t \to T$.*
>
> **Contractivity and Comparison Principle:** *Let $T \in (0, \infty]$ and f_1, $f_2 \in \mathcal{C}([0,T), (L^p_\ell \cap L^1)(\mathbb{R}^2))$ be two solutions to the BEFP equation (2) with initial data $f_{1,0}$ and $f_{2,0}$ in $(L^p_\ell \cap L^1)(\mathbb{R}^2)$, respectively. Then*
> $$\|f_1(t) - f_2(t)\|_{L^1} \leq \|f_0 - g_0\|_{L^1} , \quad t \in [0,T)$$
> *Furthermore, if $f_{1,0} \leq f_{2,0}$ then $f_1(t,v) \leq f_2(t,v)$ for all $t \in [0,T)$ and $v \in \mathbb{R}^2$.*

Thus one can deduce the following result from Theorem 3.3 and Proposition 4.1:

Theorem 4.2. *Take a non-negative initial condition $f_0 \in L^\infty_\ell(\mathbb{R}^2)$ with $\ell > 2$. Then there is a unique solution $f \in \mathcal{C}([0,\infty), L^\infty_\ell(\mathbb{R}^2))$ (in the sense of [7]) of the BEFP equation (2) with initial condition f_0. In addition, if $\|f_0\|_{L^1_\ell} < \infty$, there is some $C > 0$ which depends only on f_0 such that*

$$\|f(t)\|_{L^1_\ell} + \|f(t)\|_{L^\infty_\ell} \leq C \quad \text{for all } t \geq 1. \tag{28}$$

Proof. The local well-posedness is provided by Proposition 4.1, so we only need to show that the norm $\|f(t)\|_{L^\infty_\ell}$ does not blow up in finite time. Let T be the maximal existence time of f given by Proposition 4.1. We consider the radially symmetric solution $f^\#$ to (2) corresponding to the initial condition

$$f^\#_0(v) = \frac{\|f_0\|_{L^\infty_\ell}}{1 + |v|^\ell} , \quad v \in \mathbb{R}^2 ,$$

which we know to be globally defined by Theorem 3.3, since $f^\#_0$ has finite mass due to the condition $\ell > 2$. Clearly, $f_0 \leq f^\#_0$ and by the comparison principle in Proposition 4.1 we know that $f(t,v) \leq f^\#(t,v)$ for all $t \in [0,T)$ and $v \in \mathbb{R}^2$. Equation (26) with $p = \infty$ gives us

$$\|f(t)\|_{L^\infty_\ell} \leq \|f^\#(t)\|_{L^\infty_\ell} \leq Ce^{2t}\|f^\#_0\|_{L^\infty_\ell}$$

for some $C > 0$ which depends only on ℓ and $\|f_0\|_{L^1}$. This estimate prevents the L_ℓ^∞-norm of $f(t)$ to explode in finite time, so the solution must in fact be globally defined, that is $T = \infty$. In order to obtain the additional statement (28) it suffices to apply equation (27) with $p = 1$ and $p = \infty$. $\qquad\qquad\square$

We now turn to the large time behavior of solutions to (2) and recall that equation (2), in any dimension d, has an entropy functional given by

$$H(f) = \int_{\mathbb{R}^d} \left[\frac{|v|^2}{2} f(v) + f(v) \log f(v) - (f(v) + 1) \log(f(v) + 1) \right] \mathrm{d}v \qquad (29)$$

in the following sense: any solution f to (2) with a bounded moment of order $\ell > 2$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f) = -D(f) := - \int_{\mathbb{R}^d} f(1 + f)(v) \left| v + \nabla \log \left(\frac{f}{1 + f} \right) (v) \right|^2 \mathrm{d}v. \qquad (30)$$

The availability of this functional leads us to the following result:

Theorem 4.3. *Take a non-negative $f_0 \in L_\ell^\infty(\mathbb{R}^2) \cap L_\ell^1(\mathbb{R}^2)$ with $\ell > 2$. Then the solution f of the BEFP equation (2) with initial condition f_0 given by Theorem 4.2 satisfies*

$$\lim_{t \to \infty} \|f(t) - f_\infty^\beta\|_{L^1} = 0 \ ,$$

where β is given by (25) in terms of the mass m of f_0.

Remark 4.4. Due to Theorem 4.2 solutions to (2) emanating from non-negative initial data in $L_\ell^\infty(\mathbb{R}^2) \cap L_\ell^1(\mathbb{R}^2)$ with $\ell > 2$ are bounded in L^∞ globally in time and a simple interpolation argument shows that for all $1 \le p < \infty$ we also have

$$\lim_{t \to \infty} \|f(t) - f_\infty^\beta\|_p = 0 \ .$$

Proof of Theorem 4.3. We follow a common technique to deduce convergence to equilibrium from the existence of a Lyapunov functional such as (29); see for example [11, 4].

Take a solution $f = f(t, v)$ of the BEFP equation (2) with initial condition f_0 and recall that Theorem 4.2 guarantees that $\{f(t) \ : \ t \ge 1\}$ is bounded in $L_\ell^\infty(\mathbb{R}^2) \cap L_\ell^1(\mathbb{R}^2)$.

Step 1: Strong convergence of $f(t_n)$ for some sequence $\{t_n\}_{n \ge 1}$ of times. From (30) (which we can use since we are assuming $f_0 \in L_\ell^1(\mathbb{R}^2)$ for $\ell > 2$) and the fact that $H(f(t))$ is bounded below by $H(f_\infty^\beta)$ (due to Lemma 4.5 below) we see that there must exist a divergent sequence $\{t_n\}_{n \ge 1}$, $t_n \ge 1$, such that $D(f(t_n)) \to 0$ as $n \to \infty$. Due to (28) we know that there exist a subsequence of $\{t_n\}_{n \ge 1}$ (which we still call $\{t_n\}$ by an abuse of notation) and a non-negative function $f_\infty \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ such that $f(t_n) \rightharpoonup f_\infty$ weakly in $L^p(\mathbb{R}^2)$, for all

$p \in [1, \infty)$, and also in the weak-$*$ topology of $L^\infty(\mathbb{R}^2)$. We also note that the bounds (28) ensure that

$$\int_{\mathbb{R}^2} f_\infty(v) \, \mathrm{d}v = \lim_{n \to \infty} \int_{\mathbb{R}^2} f(t_n, v) \, \mathrm{d}v = m := \|f_0\|_{L^1} . \tag{31}$$

Setting $f_n := f(t_n)$, $n \geq 1$, let us now see that the convergence of $\{f_n\}_{n \geq 1}$ must in fact be strong, due to the fact that $D(f_n) \to 0$. Indeed, expanding the square

$$D(f_n) = \int_{\mathbb{R}^2} f_n(1 + f_n)(v)|v|^2 \, \mathrm{d}v + \int_{\mathbb{R}^2} \frac{|\nabla f_n|^2}{f_n(1 + f_n)}(v) \, \mathrm{d}v + 2 \int_{\mathbb{R}^2} v \cdot \nabla f_n(v) \, \mathrm{d}v$$
$$\geq \int_{\mathbb{R}^2} \frac{|\nabla f_n|^2}{f_n(1 + f_n)}(v) \, \mathrm{d}v - 4m.$$

Since $D(f_n) \to 0$, this shows that the sequence $\{\log(f_n/(1+f_n))\}_{n \geq 1}$ is bounded in $H^1(\mathbb{R}^2)$. In particular, there is a subsequence of $\{f_n\}_{n \geq 1}$ (again denoted by $\{f_n\}$) which converges pointwise a.e. to f_∞ and weak limits coincide. Vitali's theorem then shows that, after possibly extracting a further subsequence, we can conclude that

$$f_n \to f_\infty \quad \text{strongly in } L^p(\mathbb{R}^2) \text{ as } n \to \infty,$$

for all $1 \leq p < \infty$.

Step 2: Proof that $f_\infty = f_\infty^\beta$. Define $h_n := f_n/(1 + f_n)$, $n \geq 1$. From the previous step, we know that h_n converges pointwise a.e. to $h_\infty := f_\infty/(1 + f_\infty)$. Since $h_n \leq f_n$ is clearly uniformly bounded in all $L^p(\mathbb{R}^2)$ ($p \in [1, \infty]$), there is another subsequence of $\{t_n\}_{n \geq 1}$ such that $\{h_n\}_{n \geq 1}$ converges weakly in all $L^p(\mathbb{R}^2)$, and weakly-$*$ in $L^\infty(\mathbb{R}^2)$ to h_∞. Now we have, by the Cauchy-Schwarz inequality,

$$\left(\int_{\mathbb{R}^2} |vh_n(v) + \nabla h_n(v)| \, \mathrm{d}v \right)^2 \leq \left(\int_{\mathbb{R}^2} h_n(v) \left| v + \frac{\nabla h_n}{h_n}(v) \right|^2 \mathrm{d}v \right) \|h_n\|_{L^1}$$
$$\leq m \, D(f_n) \to 0$$

as $n \to \infty$. This implies that $vh_n + \nabla h_n \to 0$ in $L^1(\mathbb{R}^2)$ as $n \to \infty$ which, together with the already established convergences of h_n to h_∞ implies that $vh_\infty(v) + \nabla h_\infty(v) = 0$ in the sense of distributions and in fact almost everywhere. Consequently, $h_\infty(v) = \tilde{\beta} \exp(-|v|^2/2)$ for a.e. $v \in \mathbb{R}^2$ for some $\tilde{\beta} \geq 0$ and thus $f_\infty = f_\infty^{\tilde{\beta}}$ a.e. in $\mathbb{R}^2$. Finally, $\tilde{\beta} = \beta$ is given by (25) since the mass of f_∞ is the same as that of f_0 by (31).

Step 3: Convergence of the entropy. Since $t \mapsto H(f(t))$ is nonincreasing and larger than $H(f_\infty^\beta)$ (see Lemma 4.5 below) there is $H_\infty \geq H(f_\infty^\beta)$ such that

$$H(f(t)) \to H_\infty \quad \text{as } t \to \infty. \tag{32}$$

From the previous two steps we know that $\{f_n = f(t_n)\}_{n \geq 1}$ converges pointwisely to f_∞^β while (28) ensures that $\{f_n\}_{n \geq 1}$ is bounded in $(L_\ell^1 \cap L_\ell^\infty)(\mathbb{R}^2)$, hence also

in $L^2_\ell(\mathbb{R}^2)$. Since $\ell > 2$ the Dominated Convergence Theorem now shows that $H(f_n) \to H(f^\beta_\infty)$, which implies that $H_\infty = H(f^\beta_\infty)$.

Using Lemma 4.5 we directly deduce from (32) that $\|f(t) - f^\beta_\infty\|_1 \to 0$ as $t \to \infty$, finishing the proof. $\qquad\square$

We finally give a Csiszár-Kullback inequality for our entropy functional, which was needed to show that convergence in entropy implies convergence in L^1 norm in the proof of Theorem 4.3. We would like to warmly thank the anonymous referee for suggesting an improvement of the following result in the preprint version of this paper.

Lemma 4.5 (Csiszár-Kullback-type inequality for the BEFP entropy). *Take a non-negative function $f \in L^1(\mathbb{R}^d)$ with mass $\|f\|_1 \leq m_c$, and $\beta > 0$ such that the mass of f^β_∞ is equal to that of f. Then*

$$H(f) - H(f^\beta_\infty) \geq C\|f - f^\beta_\infty\|_1^2 \tag{33}$$

for some $C > 0$ which depends only on $\|f^\beta_\infty\|_1$ and $\|f^\beta_\infty\|_2$.

Proof. We follow a well-known strategy for proving Csiszár-Kullback-type inequalities; see for example [5]. For each $v \in \mathbb{R}^d$ call

$$\phi(r) \equiv \phi(r, v) := r \log r - (r + 1) \log(r + 1) + \frac{|v|^2}{2} r, \qquad r > 0,$$

so that we have

$$H(f) = \int_{\mathbb{R}^d} \phi(f(v)) \, dv.$$

Using that $\phi'(f^\beta_\infty(v))$ is a constant independent of $v \in \mathbb{R}^d$ we can write

$$H(f) - H(f_\infty)$$
$$= \int_{\mathbb{R}^d} \left(\phi(f(v)) - \phi(f^\beta_\infty(v)) - \phi'(f^\beta_\infty(v))(f(v) - f^\beta_\infty(v)) \right) dv. \tag{34}$$

Now we notice that ϕ is a convex function with $\phi''(r) = (r(1+r))^{-1}$, so we have the bound

$$\phi(r) - \phi(s) - \phi'(s)(r - s) \geq \frac{(r - s)^2}{s(1 + s)} \geq 0$$

whenever $0 < r \leq s$. Using this in (34) and denoting $\Omega := \{v \in \mathbb{R}^d \mid f(v) \leq f^\beta_\infty(v)\}$ we obtain (writing for convenience $f \equiv f(v)$ and $f^\beta_\infty \equiv f^\beta_\infty(v)$ inside the integral)

$$H(f) - H(f_\infty) \geq \int_\Omega \left(\phi(f) - \phi(f^\beta_\infty) - \phi'(f^\beta_\infty)(f - f^\beta_\infty) \right) dv$$
$$\geq \int_\Omega \frac{(f - f^\beta_\infty)^2}{f^\beta_\infty(1 + f^\beta_\infty)} \, dv \geq \left(\int_\Omega (f^\beta_\infty - f) \, dv \right)^2 \left(\int_{\mathbb{R}^d} f^\beta_\infty(v)(1 + f^\beta_\infty) \, dv \right)^{-2},$$

where for the last step we used the Cauchy-Schwarz inequality. Calling

$$C := \frac{1}{4} \left(\int_{\mathbb{R}^d} f_\infty^\beta (1 + f_\infty^\beta) \, \mathrm{d}v \right)^{-2}$$

and observing that

$$\int_\Omega (f_\infty^\beta - f) \, \mathrm{d}v = \frac{1}{2} \| f_\infty^\beta - f \|_1$$

this gives precisely

$$H(f) - H(f_\infty) \geq C \| f - f_\infty^\beta \|_1^2.$$

$\square$

References

[1] A. Aftalion and B. Helffer. On mathematical models for Bose-Einstein condensates in optical lattices. *Rev. Math. Phys.*, 21(2):229–278, 2009.

[2] T. Allemand and G. Toscani. The grazing collision limit of Kac caricature of Bose-Einstein particles. *Asymptot. Anal.*, 72(3-4):201–229, 2011.

[3] N. Ben Abdallah, I. M. Gamba, and G. Toscani. On the minimization problem of sub-linear convex functionals. *Kinet. Relat. Models*, 4(4):857–871, 2011.

[4] D. Benedetto, E. Caglioti, J. A. Carrillo, and M. Pulvirenti. A non-Maxwellian steady distribution for one-dimensional granular media. *J. Statist. Phys.*, 91(5-6):979–990, 1998.

[5] M. J. Cáceres, J. A. Carrillo, and J. Dolbeault. Nonlinear Stability in L^p for a Confined System of Charged Particles. *SIAM Journal on Mathematical Analysis*, 34(2):478–494, Jan. 2002.

[6] J. A. Carrillo, S. Cordier, and G. Toscani. Over-populated tails for conservative-in-the-mean inelastic Maxwell models. *Discrete Contin. Dyn. Syst.*, 24(1):59–81, 2009.

[7] J. A. Carrillo, Ph. Laurençot, and J. Rosado. Fermi-Dirac-Fokker-Planck equation: well-posedness & long-time asymptotics. *J. Differential Equations*, 247(8):2209–2234, 2009.

[8] J. A. Carrillo, J. Rosado, and F. Salvarani. 1D nonlinear Fokker–Planck equations for fermions and bosons. *Appl. Math. Lett.*, 21(2):148–154, 2008.

[9] J. A. Carrillo and G. Toscani. Exponential convergence toward equilibrium for homogeneous Fokker-Planck-type equations. *Math. Methods Appl. Sci.*, 21(13):1269–1286, 1998.

[10] S. Cordier, L. Pareschi, and G. Toscani. On a kinetic model for a simple market economy. *J. Statist. Phys.*, 120(1-2):253–277, 2005.

[11] L. Desvillettes and J. Dolbeault. On long time asymptotics of the Vlasov-Poisson-Boltzmann equation. *Comm. Partial Differential Equations*, 16(2-3):451–489, 1991.

[12] L. Desvillettes and V. Ricci. A rigorous derivation of a linear kinetic equation of Fokker-Planck type in the limit of grazing collisions. *J. Statist. Phys.*, 104(5-6):1173–1189, 2001.

[13] L. Erdös, B. Schlein, and H. T. Yau. Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate. *Ann. of Math.*, 172:291–370, 2010.

[14] M. Escobedo and S. Mischler. On a quantum Boltzmann equation for a gas of photons. *J. Math. Pures Appl.*, 80(5):471 – 515, 2001.

[15] M. Escobedo, S. Mischler, and J. Velazquez. Asymptotic description of Dirac mass formation in kinetic equations for quantum particles. *J. Differential Equations*, 202(2):208 – 230, 2004.

[16] A. D. Fokker. Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld. *Ann. Phys.*, 43:810–820, 1914.

[17] G. W. Ford, J. T. Lewis, and R. F. O'Connell. Quantum Langevin equation. *Phys. Rev. A*, 37:4419–4428, 1988.

[18] T. Frank. *Nonlinear Fokker-Planck Equations: Fundamentals and Applications.* Springer Series in Synergetics. Springer, 2006.

[19] T. D. Frank. Classical Langevin equations for the free electron gas and blackbody radiation. *J. Phys. A*, 37:3561–3567, 2004.

[20] I. M. Gamba and J. R. Haack. A conservative spectral method for the Boltzmann equation with anisotropic scattering and the grazing collisions limit. *J. Comput. Phys.*, 270:40–57, 2014.

[21] C. W. Gardiner. *Handbook of stochastic methods for physics, chemistry and the natural sciences*, volume 13 of *Springer Series in Synergetics.* Springer-Verlag, Berlin, third edition, 2004.

[22] C. W. Gardiner and M. J. Collett. Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation. *Phys. Rev. A*, 31:3761–3774, 1985.

[23] T. Goudon. On Boltzmann equations and Fokker-Planck asymptotics: influence of grazing collisions. *J. Statist. Phys.*, 89(3-4):751–776, 1997.

[24] B. L. Hu, J. P. Paz, and Y. Zhang. Quantum Brownian motion in a general environment: Exact master equation with nonlocal dissipation and colored noise. *Phys. Rev. D*, 45:2843–2861, 1992.

[25] G. Kaniadakis. Generalized Boltzmann equation describing the dynamics of bosons and fermions. *Phys. Lett. A*, 203:229–234, 1995.

[26] G. Kaniadakis and P. Quarati. Kinetic equation for classical particles obeying an exclusion principle. *Phys. Rev. E*, 48:4263–4270, 1993.

[27] X. Lu. On spatially homogeneous solutions of a modified Boltzmann equation for Fermi–Dirac particles. *J. Statist. Phys.*, 105(1-2):353–388, 2001.

[28] M. Planck. Über einen Satz der statistischen Dynamik und seine Erweiterung in der Quantentheorie. *Sitz. ber. Preuss. Akad. Wiss. (Berlin)*, pages 324–341, 1917.

[29] H. Risken. *The Fokker-Planck equation*, volume 18 of *Springer Series in Synergetics.* Springer-Verlag, Berlin, second edition, 1989.

[30] A. Rossani and G. Kaniadakis. A generalized quasi-classical Boltzmann equation. *Physica A*, 277:349–358, 2000.

[31] J. Sopik, C. Sire, and P.-H. Chavanis. Dynamics of the Bose-Einstein condensation: analogy with the collapse dynamics of a classical self-gravitating Brownian gas. *Phys. Rev. E*, 74:011112, 2006.

[32] K. Staliunas. Gross-Pitaevskii model for nonzero temperature Bose-Einstein condensates. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 9(16):2713–2719, 2006.

[33] G. Toscani. Finite time blow up in Kaniadakis-Quarati model of Bose-Einstein particles. *Comm. Partial Differential Equations*, 37(1):77–87, 2011.

[34] R. Yano. On quantum Fokker–Planck equation. *J. Statist. Phys.*, 158(1):231–247, 2015.

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